Generating Random Points in a Tetrahedron

C. Rocchini, P. Cignoni*
Istituto di Elaborazione dell'Informazione – Consiglio Nazionale delle Ricerche
CNR Research Park, Via V. Alfieri 1 - 56010 S.Giuliano (Pisa), ITALY

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Abstract

This paper proposes a simple and efficient technique to generate uniformly random points in a tetrahedron. The technique generates random points in a cube, and folds the cube into the barycentric space of the tetrahedron in a way that preserves uniformity.

1 Introduction

The problem of picking a random point into a tetrahedron with uniform distribution can arise in various situation, typically in Scientific Visualization when managing unstructured datasets represented by tetrahedral meshes. In some applications [1] there can be the need of creating more than $O(10^7)$ samples, so an efficient technique to generate a random sample inside a tetrahedron can be very useful.

The method proposed here is a generalization of one of the technique used by Turk in [2] to generate a random point inside a triangle. The main idea of his 2D technique is to generate a random point in a parallelogram and reflect it around the center of the parallelogram. In this paper we extend this approach to 3D, generating random points with uniform distribution in a cube, and folding them into a tetrahedron. Source code of an implementation of this technique is available online.

2 Folding a cube into a tetrahedron

Let s, t and u be three numbers chosen form a uniform distribution of random numbers in the interval [0,1] and $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ three 3D vectors; the point $s\mathbf{a} + t\mathbf{b} + u\mathbf{c}$ identifies a random point with uniform distribution inside the 3D skewed parallelepiped defined by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. To simplify the discussion we will focus

*{rocchini,cignoni}@iei.pi.cnr.it http://vcg.iei.pi.cnr.it

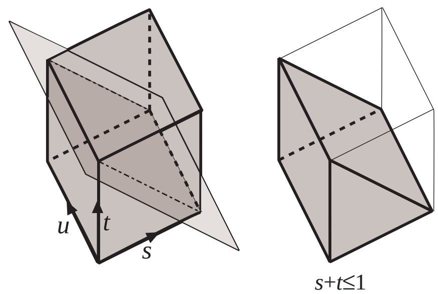

Figure 1: The plane $s + t = 1$ cut the cube into two equal volume triangular prisms.

only onto s, t and u and how how to *dissect and fold* the cubic parametric space defined by s, t and u into a tetrahedron.

It worth to be noted that while the area of a triangle is 1/2 of the area of the corresponding parallelogram, the volume of a tetrahedron is 1/6 of the volume of the corresponding parallelepiped, therefore we will dissect and fold it in two and the result again in three parts in order to obtain the $2 \times 3 = 6$ tetrahedra.

The first step is to cut the cube with the plane $s + t = 1$ into two triangular prism of equal volume, and then folding all the points falling beyond the plane $s + t = 1$ (i.e. in the upper prism) into the lower prism as shown in Figure 1; this can be done by calculating the new (s, t, u) values as shown in Equation 1.

$$(s,t,u) = \begin{cases} (s,t,u) & \text{if } s+t \leq 1 \\ (1-s, 1-t, u) & \text{if } s+t > 1 \end{cases} \quad (1)$$

The second step is to cut and fold the resulting triangular prism with the two planes $t + u = 1$ and $s + t + u = 1$. This dissection identifies the three equal volume tetrahedra shown in Figure 2; the folding of the triangular prism into the first tetrahedron can be done by calculating the new (s, t, u) values as shown in Equation 2.

$$(s,t,u) = \begin{cases} (s,t,u) & \text{if } s+t+u \leq 1 \\ \begin{cases} (s, 1-u, 1-s-t) & \text{if } t+u > 1 \\ (1-t-u, t, s+t+u-1) & \text{if } t+u \leq 1 \end{cases} & \text{if } s+t+u > 1 \end{cases} \quad (2)$$

3 Alternative Approach

A technique to generate a random point inside a tetrahedron is described in [2]: three random numbers are chosen from a uniform distribution; the cube root of the first number is used to pick a triangle that is parallel to the base of the tetrahedron, and then the two remaining numbers are used to pick a random

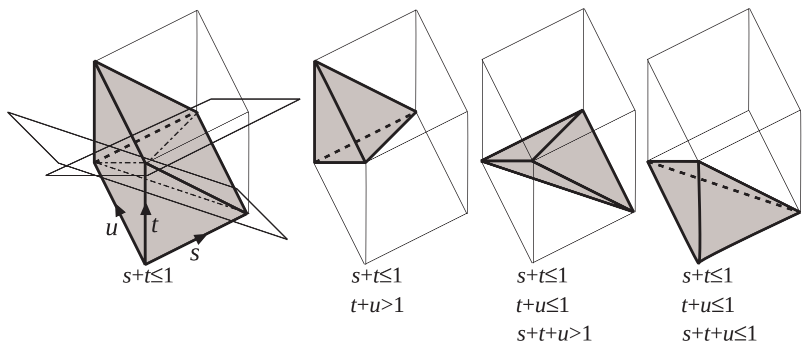

Figure 2: The triangular prism can be cut into three equal volume tetrahedra by the planes $t + u = 1$ and $s + t + u = 1$. Below each tetrahedron we show the inequalities that are satisfied by the points it belongs.

point on that triangle. This technique is approximately twice slower than the one proposed here, but, on the other hand, it allows stratification. Another of the advantages of our technique is that the barycentric coordinates of the sampled point are available at the end of the process; therefore it is easy to find interpolate values for attributes of the sampled point that were originally defined onto the tetrahedron vertexes.

References

[1] P. Cignoni, C. Rocchini, and R. Scopigno. Metro 3d: Measuring error on simplified tetrahedral complexes. Technical Report B4-09-00, I.E.I. – C.N.R., Pisa, Italy, Febr. 2000.

[2] Greg Turk. Generating random points in triangles. In A. S. Glassner, editor, *Graphics Gems*, pages 24–28. Academic Press, 1990.

Web Information
Source code and additional info are available online at
http://www.acm.org/jgt/papers/RocchiniCignoni00